The semester consists of n lectures

You need to split it into two parts

s.t. there is 1 exam in the first and two exams in the second.

How many ways to do this?

$$a_n = \sum_{k=0}^{n} k \cdot \binom{n-k}{2}$$

Let $F(x)$ be the gen. function for a_n.

In tis case

$$F(x) = \sum_{n\geq 0} \left(\sum_{k=0}^{n} k \binom{n-k}{2} x^n \right)$$

$$F(x) = \sum_{n \geq 0} \left(\sum_{k=0}^{n} k \binom{n-k}{2} x^{k} \cdot x^{n-k} \right)$$

Let $G(x) = \sum_{n \geq 0} b_n x^n$ and $H(x) = \sum_{n \geq 0} c_n x^n$

Then the formula for $(G \cdot H)(x) = \sum_{n \geq 0} \left(\sum_{i=0}^{n} b_i c_{n-i} x^n \right)$

For example, Let $G(x) = 1 + x^2$ $H(x) = 2 + 3x + x^3$

$$(G \cdot H)(x) = ((1+x^2)(2+3x+x^3)) = 1 \cdot 2 + (1 \cdot 3x + 0 \cdot 2x) + (1 \cdot 0x^2 + 0 \cdot 3x^2 + 1 \cdot 2x^2) + (1 \cdot 1x^3 + 0 \cdot 0x^3 + 1 \cdot 3x^3 + 0 \cdot 2x^3) + \ldots$$

$$F(x) = \left(\sum_{k \geq 0} k x^k \right) \cdot \left(\sum_{\ell \geq 0} \binom{\ell}{2} x^{\ell} \right)$$

$x^n = x^i \cdot x^{n-i}$

$$F(x) = \left(\sum k\, x^k\right) \left(\sum \binom{\ell}{2} x^\ell\right)$$

Note that $\displaystyle \sum k x^k = x \cdot \frac{d \sum x^k}{dx} = x \cdot \frac{d}{dx}\left(\frac{1}{1-x}\right)$

$$= \frac{x}{(1-x)^2}$$

$$\sum \binom{\ell}{2} x^\ell = \frac{1}{2} \sum_{\ell \geq 0} \ell(\ell-1)\, x^\ell = \frac{1}{2} x^2 \frac{d^2 \sum x^\ell}{dx^2} =$$

$$= \frac{x^2}{2} \frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = \frac{x^2}{\cancel{2}} \frac{\cancel{2}}{(1-x)^3} = \frac{x^2}{(1-x)^3}$$

$$F(x) = \frac{x}{(1-x)^2} \cdot \frac{x^2}{(1-x)^3} = x^3 \cdot \frac{1}{(1-x)^5}$$

Exercise

Find a sequence b_n s.t. it's gener. function is $\frac{4!}{(1-x)^5} = \frac{d^4 \frac{1}{1-x}}{dx} =$

$$= \frac{d^4}{dx} \sum_{k\geq 0} x^k = \sum_{k\geq 4} k(k-1)(k-2)(k-3)\cdot x^{k-4} =$$

$$= \sum_{k\geq 0} (k+4)(k+3)(k+2)(k+1)\, x^k$$

$$F(x) = \sum_{k\geq 0} \binom{k+4}{4} x^{k+3}$$

Therefore $a_{k+3} = \binom{k+4}{4}$. So $a_k = \binom{k+1}{4}$ for $k \geq 3$ and $a_k = 0$ other.