- To any seq. $\underbrace{w_1 \dots w_\ell}_{\in [n]} \rightsquigarrow p_1 \dots p_k \in S_n$

- We defined $\mathcal{D}(p,q)$ s.t.

$$n + \sum_{i=1}^{k-1} \mathcal{D}(p_i, p_{i+1}) \leq \ell$$

We proved

$\ell \geq n + n! - 1$: $\mathcal{D}(p_1 \dots p_t) = \sum_{i=1}^{t-1} \mathcal{D}(p_i, p_{i+1}) \geq C_0(p_1 \dots p_k) - 1$

where $C_0(p_1 \dots p_k)$ is the number of dif. permutations in $p_1 \dots p_k$

$l \geq n! + (n-1)! + (n-2)$:

We need to introduce a new notion of

1-cycle class

Example of a 1-cycle class

$\{12345, 23451, 34512, 45123, 51234\}$

so $\{p_1 \ldots p_n\} \subseteq S_n$ is a 1-cycle class

iff $p_{j+1}(n) = p_j(1)$ and $p_{j+1}(1) = p_j(2) \ldots$

Another example

$\{231, 312, 123\}$

$\{321, 213, 132\}$

There are $(n-1)!$ 1-cycle classes.

We prove that

$$D(p_1 \dots p_t) \geq C_0(p_1 \dots p_t) + C_1(p_1 \dots p_t) - 1$$

where $C_1(p_1 \dots p_t)$ is the number of complete 1-cycle classes in $p_1 \dots p_{t-1}$

(a class $\{q_1 \dots q_n\}$ is complete in $p_1 \dots p_{t-1}$ iff for any $i \in [n]$ there is $j \in [t-1]$ s.t. $q_i = p_j$)

$\{123, 231, 312\}$ - t.c. c. it's complete

in

123 132 231 312

it's not complete in 123 132 312

$$\mathcal{D}(p_1 \dots p_t) \geq C_0(p_1 \dots p_t) + C_1(p_1 \dots p_t) - 1 \quad (1)$$

Note that:

$$C_0(p_1 \dots p_{t+1}) \leq C_0(p_1 \dots p_t) + 1$$

$$C_1(p_1 \dots p_{t+1}) \leq C_1(p_1 \dots p_t) + 1$$

So if $\mathcal{D}(p_t, p_{t+1}) \geq 2$, then the ineq. (1) is true.

If $\mathcal{D}(p_t, p_{t+1}) = 1$, then p_t and p_{t+1} are in the same cycle.

If the cycle cont. p_t and p_{t+1} is complete in $p_1 \dots p_t$, then C_0 doesn't increase

But if the cycle is not complete C_1 doesn't increase.

In the end:

$$C_0(p_1 \dots p_k) = n!$$

$$C_1(p_1 \dots p_k) \geq (n-1)! - 1$$

Therefore

$$\ell \geq n + n! + (n-1)! - 1 - 1$$

$$= n! + (n-1)! + (n-2)$$

for example

123 132 231 213 312 | 321

not completed the cycle

{321, 213, 132}