Definition

A sequence $w_1 \dots w_\ell$ is an n-superperm. iff $\forall p \in S_n$ there is $i \in [\ell - n]$ s.t.

$$p(1) = x_{i+1} \;\dots\; p(n) = x_{i+n}$$

Theorem

Every n-superperm. has length $(\ell) \geq$

$$n! + (n-1)! + (n-2)! + (n-3) \cdot$$

Proof

Let $p, q \in S_n$, then $D(p,q)$ is the minimal number of symbols you need to append to p to get q.

$p = 123 \qquad q = 231$

$$\underbrace{1\,\rlap{$\overbrace{}^{q}$}2\,3}_{p}\,1 \Rightarrow D(p,q) = 1$$

$p = 123 \qquad q = 132$

$$\underbrace{1\,2\,3}_{p}\,\overbrace{1\,3\,2}^{q}$$

perm

$$D(p,q) = 3$$

$\underbrace{p(1)\,p(2) \ldots p(n)\,s}$

$\Downarrow$

so $s = p(1)$

Therefore $\forall p \in S_n^d$

there is only 1 perm. $q \in S_n^d$

s.t. $D(p, q) = 1$

$p(1)\,p(2)\,\underbrace{p(3) \ldots p(n)\,s_1\,s_2}_{q}$

$q = p(3) \ldots p(n)\,p(2)\,p(1)$

Since $p(2) \ldots p(n)\,s_1$

is not a perm

and $s_1, s_2 \in \{p(1), p(2)\}$

We say that $\mathcal{D}(p,q) = k$ iff
there are $v_1 \dots v_k$ s.t.

$$p(k+1) \dots p(n)\, v_1 \dots v_k = q$$

But

$p(i) \dots p(n)\, v_1 \dots v_{i-1}$ is not a permut.

$p = 123$ $q = 312$ $\mathcal{D}(p,q)$

$p = 123 \qquad q = 321$

$$\overbrace{\underbrace{12\,3\,2}_{\notin S_n}\,1}^{q}$$

$$123\,2\,1 \qquad \text{So } D(p,q) = 2$$

$$123 \to 232 \to 321$$

Let $w_1 \dots w_\ell$ be a super perm.

We know that $\underbrace{\{p \in S_n : \exists i \; p(1) = w_{i+1} \dots p(n) = w_{i+n}\}}_{=\, S_n}$

$$12321 \rightsquigarrow \{123, 321\}$$

Let $p_1 \dots p_k$ be the perm. occur. in the sequence

For example, 123211132 $\rightsquigarrow$ $\underbrace{123}_{p_1}$, 321, 132

123123 $\rightsquigarrow$ 123, 231, 312, 123

Note that $n + \sum_{i=1}^{k-1} \phi(p_i, p_{i+1}) \leq \ell$

length of our sequence.

We want to prove that $\ell \geq n! + (n-1)$

Consider $C_0(p_1 \cdots p_n) = \#$ dif. perm occur in $w_1 \ldots w_k$

$C_0(123) = 1$ $\quad C_0(123, 231) = 2$

Note that $C_0(p_1 \ldots p_k) = n!$ and

$C_0(p_1) = 1$

We claim that $D(p_1 \ldots p_m) \geq C_0(p_1 \ldots p_m) - 1$

$$\sum_{i=1}^{m-1} D(p_i, p_{i+1})$$

It's clear that

$$D(p_1 \cdots p_m) + 1 \leq D(p_1 \cdots p_{m+1})$$

and

$$C_0(p_1 \cdots p_{m+1}) \leq C_0(p_1 \cdots p_m) + 1$$

Note that $C_0(p_1 \cdots p_k) = n!$

So $D(p_1 \cdots p_k) \geq n! - 1$

Therefore $\ell \geq n! - 1 + n = n! + (n-1)$