Name: ______________________

Pid: ______________________

1. (10 points) Prove that there is an undecidable set W such that the sets $\{x : (n, x) \in W\}$ and $\{x : (x, n) \in W\}$ are decidable for all $n \in \mathbb{N}$

Solution: Let us consider the set $W = \{(n, n) : U(n, n) = 1\}$. It is easy to see that the set is undecidable. However, the sets $\{x : (n, x) \in W\}$ and $\{x : (x, n) \in W\}$ are all finite sets (they have either 0 or 1 element); hence, they are decidable.

2. (10 points) Let $U : \mathbb{N}^2 \to \mathbb{N}$ be a Gödel universal function. Prove that there is $p \in \mathbb{N}$ such that

$$U(p, x) = \begin{cases} 1 & \text{if } x = p^2 \\ 0 & \text{otherwise} \end{cases}.$$

Solution: Let $V(n, x)$ be an algorithm such that

$$V(n, x) = \begin{cases} 1 & \text{if } x = n^2 \\ 0 & \text{otherwise} \end{cases}.$$

Since U is a Gödel universal function there is a total $s : \mathbb{N} \to \mathbb{N}$ such that $U(s(n), x) = V(n, x)$ for all $n, x \in \mathbb{N}$. Bote that by Klene's theorem, there is $n_0 \in \mathbb{N}$ such that $U(n_0, x) = U(s(n_0), x) = V(n_0, x)$ for all $x \in \mathbb{N}$; i.e., there is $n_0 \in \mathbb{N}$ such that

$$U(n_0, x) = \begin{cases} 1 & \text{if } x = n_0^2 \\ 0 & \text{otherwise} \end{cases}.$$