Lecture 6: Structural Induction and Relations (?)

Let $S \subseteq \mathbb{R}$ s.t. $1 \in S$ and $x \in S \Rightarrow (x+1) \in S$ for any $x \in \mathbb{N}$

Theorem Let $B \subseteq U$, and $\mathcal{F} = \{ f_1 : U^{l_1} \to U \ldots \}$

Let S be the set generated by $\mathcal{F}$ from B

Consider $S' \subseteq U$ s.t.

- $B \subseteq S'$
- $f_i(u_1, \ldots u_{l_i}) \in S'$ for any $u_1, \ldots u_{l_i} \in S'$

Then $S \subseteq S'$.

Theorem

For any binary tree T, $s(T) < 2^{h(T)}$.

Proof

Let S be the set of all binary trees.

Let $S' = \{ T \in S : s(T) \leq 2^{h(T)} \}$.

We want to prove that $S' = S$ so we need to prove that $S' \supseteq S$.

- If B is the set of trees made of one integer $B \leq S'$
- Let $T = (T_1\ T_2)$ s.t. $T_1, T_2 \in S'$

$$s(T) = s(T_1) + s(T_2) \leq 2^{h(T_1)} + 2^{h(T_2)} \leq 2 \cdot 2^{\max(h(T_1), h(T_2))}$$

$$h(T) = \max(h(T_1), h(T_2)) + 1$$

$$\parallel$$

$$2^{h(T)}$$

So $T \in S'$. Hence, by the str. Ind. principle $S' \supseteq S$; i.e., $S = S'$.

Theorem Let $B \subseteq U$, and $\mathcal{F} = \{ f_i : U^{\ell_i} \to U \dots \}$

Let S be the set generated by $\mathcal{F}$ from B

Consider $S' \subseteq U$ s.t.

- $B \subseteq S'$
- ✓ $f_i(u_1, \dots u_{\ell_i}) \in S'$ for any $u_1, \dots u_{\ell_i} \in S'$

Then $S \subseteq S'$.

S is the set generated by $\mathcal{F}$ from B iff

$u \in S$ iff $\exists u_1, \dots u_m \in U$ $u_m = u$

s.t. $\forall i \in [m]$

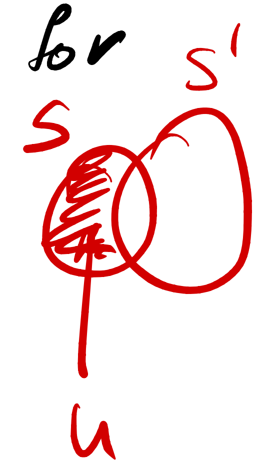
- $u_i \in B$
- $u_i = f_j(u_{i_1}, \dots u_{i_{\ell_j}})$ where $i_1, \dots i_{\ell_j} < i$

P(m):

"If for $u \in U$ there are $u_1 \dots u_m$ s.t.
$u = u_m$
for any $i \in [m]$
- either $u_i \in B$,
- or $u_i = f(u_{i_1} \dots u_{i_\ell})$ where $f \in \mathcal{F}$, $i_1 \dots i_\ell < i$

then $u \in S'$"

If we prove that P(m) is true for any $m \in \mathbb{N}$, we prove the statement.



Indeed, assume that $u \in S \setminus S'$, in this case there are $u_1 \dots u_n$ s.t. $u_n = u$ and $u_i \in B$ or $u_i = f(u_{i_1} \dots u_{i_\ell})$ for $f \in \mathcal{F}$ $i_1 \dots i_\ell < i$.

But P(n) is true, so $u \in S'$ which is a contradiction.

$P(1)$ says that if

"if $u \in U$ and there is $u_1 \in U$ s.t. $u_1 = u$ and $u_1 \in B$, $u \in S'$"

which is true since $B \subseteq S'$.

Assume that $P(m)$ is true.

consider $u \in U$ s.t. there are $u_1 \dots u_{m+1}$ "u"

- if $u_{m+1} \in B$ "u", then $u \in B \subseteq S'$
- otherwise $u_{m+1} = f(u_{i_1} \dots u_{i_e})$

Note that $u_{i_1} \dots u_{i_e} \in S'$

so $u_{m+1} = u \in S'$