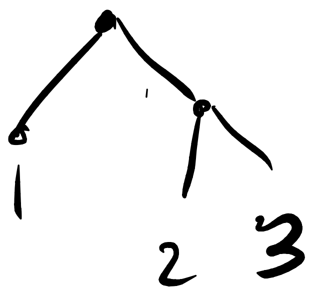
Lecture 5: Structural Induction

(1 (2 3))



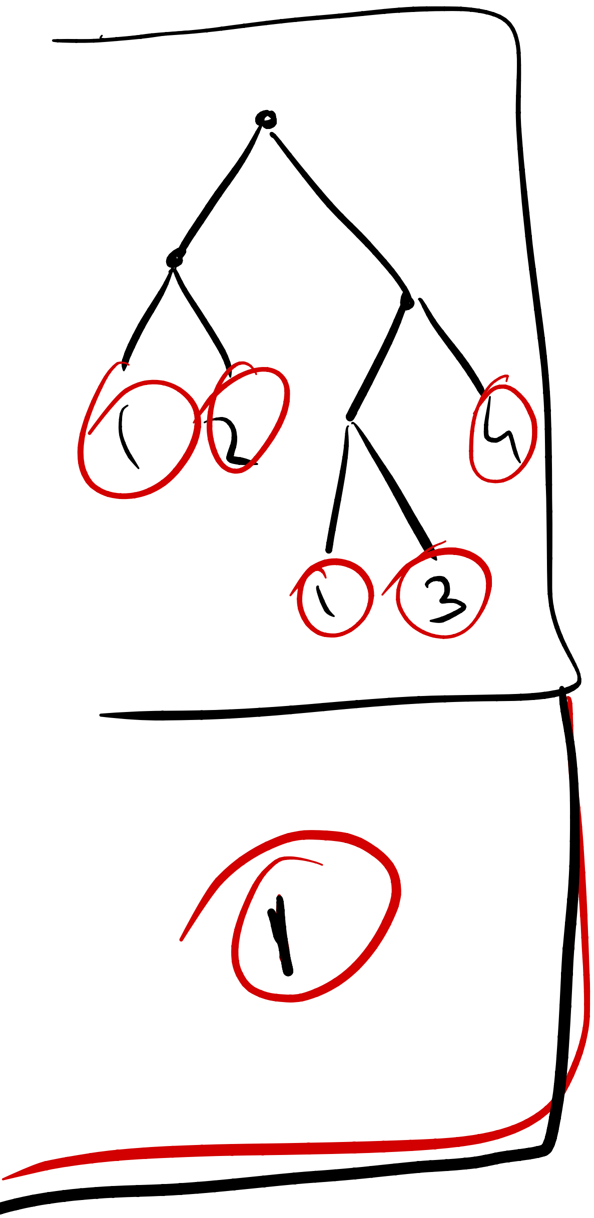
$$h((T_1, T_2)) = \max(h(T_1), h(T_2)) + 1$$

$$h(i) = 0$$

(1 1)

1

Exercise Define „size" of a binary tree.



Definition Let T be a binary tree

(base case) If T is an integer, then $s(T)=1$ and $h(T)=0$

(recursion step) Let $T=(T_1, T_2)$. Then

$$s(T) = s(T_1)+s(T_2) \text{ and}$$

$$h(T) = \max(h(T_1), h(T_2))+1$$

Exercise

What is the height and size of

$(1\ (2,3))$ and $((1\,2)\,(1\,2))$

$$h((1\ (2\ 3))) = \max\{h(1), h((2\ 3))\}+1$$

$$h\Big((1\ (2\ 3))\Big) = \max\{h(1), h((2\ 3))\}+1 =$$

$$= \max\{0, \max\{h(2), h(3)\}+1\}+1 =$$

$$= \max\{0, \max\{0, 0\}+1\}+1 =$$

$$= \max\{0, 1\}+1 = 1+1 = 2$$

$$s\Big((1\ (2\ 3))\Big) = s(1) + s((2,3)) = s(1) +$$

$$+ s(2) + s(3) = 3$$

Let $U = \mathbb{R}$, $B = \{0\}$, and $\mathcal{F} = \{f, g\}$

$f(x, y) = x + y$
$g(x) = x$

Consider $v : S \to \mathbb{R}$ s.t.
- $v(0) = 0$
- $v(f(x, y)) = f(v(x), v(y))$

$v(g(x)) = v(x) + 1$

$f(0, 0) = 0$ $\quad v(0) = v(f(0, 0)) = f(\underset{\underset{f(0,0)}{\parallel}}{v(0)}, v(0))$

$f(0,0) \downarrow 0$

$v(\underset{\underset{v(0)}{\parallel}}{g(0)}) = v(0) + 1 = 1$

Definition

A set S is freely generated by $\mathcal{F}$ from B iff

- S is generated by $\mathcal{F}$ from B and
- $B \cap \mathrm{Im} f \overset{=\emptyset}{,}$ $\mathrm{Im} f \cap \mathrm{Im}\, g = \emptyset$ for any $f, g \in \mathcal{F}$

Theorem Let $S \subseteq U$ be the set freely generated from B by $\mathcal{F} = \{ f_1 : U^{l_1} \to U \dots f_n : U^{l_n} \to U \}$

Let $F_B : B \to V$, $F_1 : U^{l_1} \to V \dots$ $F_n : U^{l_n} \to V$

Then there is a function $h : S \to V$ s.t.

- $h(u) = F_B(u)$ for all $u \in B$
- $h(f_i(u_1 \dots u_{l_i})) = F_i(h(u_1) \dots h(u_{l_i}))$

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$$h(T) = \max(h(T_1), h(T_2)) + 1$$

Exercise: Define a function that sums up all the numbers in the tree

$$s(T) \leq 2^{h(T)}$$