# Lecture 19

Let's conside a signature $\mathcal{S}$ and a structure $\mathcal{M}$. Consider a formula $\varphi$ in $\mathcal{S}$

For example $\mathcal{S}' = (\leq ; +)$

$$\mathcal{M} = (\mathbb{N} ; \leq , +) \qquad \mathcal{M} = (\mathbb{N} ; \mid , \times)$$

$$\varphi = \forall x \, \forall y \; \underbrace{(x \leq y)}_{\substack{xk = y \\ \text{for some } k}} \Rightarrow \underbrace{(x + x \leq y + y)}_{\substack{\text{this says that } x^2 \ell = y^2 \\ \text{for some } \ell}}$$

Is $\varphi$ true in $\mathcal{M}$?

Terms are

- variables
- functions applied to terms

$\mathcal{S}' = ( \leq ; + )$

$\mathcal{M} = ( \mathbb{N} ; \leq , + )$

$\varphi = \forall x \, \forall y \; (x \leq y) \Rightarrow (x + x \leq y + y)$

$x_1 \qquad x_1 + x_2$

$(x_1 + x_2) + x_3$

$\vdots$

$+ (+ \, x_1 \, x_2) \; x_3$

Let $\Omega$ be the set of variables.

an assignement $s : \Omega \to \mathbb{N}$

defines $\mathcal{M} \models \varphi[s]$ it's true with as. $s$ to the variables in structure $\mathcal{M}$.

Let $t$ be a term in $\mathcal{S}$ and $s:\Omega \to \underline{M}$

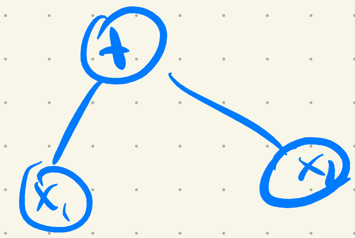
- if $t = x_i$, then $t^{\mathcal{M}}[s] = s(x_i)$
- if $t = f(t_1 \ldots t_k)$ then $t^{\mathcal{M}}[s] = f^{\mathcal{M}}(t_1^{\mathcal{M}}[s] \ldots t_k^{\mathcal{M}}[s])$

$\mathcal{M} = \langle \underline{M}, F_{rel}, F_{fun} \rangle$

Let $\mathcal{M} = (\mathbb{N}; \leq, +)$

consider $t = x_1 + x_2$

$s(x_i) = 2$




then $t^{\mathcal{M}}[s] =$

if $s(x_1) = 0$ and $s(x_2) = 10$

then $t^{\mathcal{M}}[s] = 10$

Atomic formulas

- $P(t_1 \dots t_k)$, where $P$ is a predicate in $S$

in case of $S = (\leq, +)$ and $\mathcal{M} = (\mathbb{N}; \leq; +)$

atomic formulas are of the form

$$((x_1 + x_2) + \dots x_n) \leq ((y_1 + y_2) + \dots y_k)$$

let $\alpha$ be an atomic formula, then

$\mathcal{M} \models \alpha[s]$ iff $P^{\mathcal{M}}(t_1^{\mathcal{M}}[s] \dots t_k^{\mathcal{M}}[s])$

Consider $s(x_i) = 2$ for all $i$

and $\alpha = (x_1 + x_2) \leq x_3$

$\mathcal{M} = (\mathbb{N}; \leq; +)$

$\mathcal{M} \not\models \alpha[s]$

Formula

$\varphi$ is a formula if

- $\varphi = (\psi_1 \wedge \psi_2)$, where $\psi_i$'s are formulas
- $\varphi = (\psi_1 \Rightarrow \psi_2)$
- $\varphi = \neg \psi$
- $\varphi = (\psi_1 \vee \psi_2)$
- $\varphi = \exists x_i \; \psi$
- $\varphi = \forall x_i \; \psi$
- $\varphi = \alpha$, where $\alpha$ is an atomic formula.

Let $\varphi$ be a formula then

$\mathcal{M} \models \varphi[s]$ iff

- if $\mathcal{M} \models \psi_1[s]$ and $\mathcal{M} \models \psi_2[s]$ and $\varphi = \psi_1 \wedge \psi_2$
- if $\mathcal{M} \not\models \psi_1[s]$ or $\mathcal{M} \models \psi_2[s]$ and $\varphi = \psi_1 \Rightarrow \psi_2$
- if $\mathcal{M} \not\models \psi[s]$ and $\varphi = \neg \psi$
- if $\mathcal{M} \models \psi_1[s]$ or $\mathcal{M} \models \psi_2[s]$ $\varphi = \psi_1 \vee \psi_2$
- if $\mathcal{M} \models \psi[s(x_i|v)]$ for $v \in M$ $\varphi = \exists x_i \, \psi$

  and $s(x_i|v)(y) = \begin{cases} s(y) & \text{if } y \neq x_i \\ v & \text{if } y = x_i \end{cases}$

- if $\mathcal{M} \models \psi[s(x_i|v)]$ for all $v \in M$

- for atomic formulas we use the prev. def.

$\exists x \, \forall y \quad (x+y) \leq x$

```
int x=2
if (....) {
    int x=3
    print(x)
}
```