Lecture 11: Prop. formulas, Truth Tables, Semantic Implications

Theorem

Any Boolean function $f : \{T, F\}^n \to \{T, F\}$ has a prop. formula repr. φ; i.e.,

$$\varphi\big|_{x_1 = v_1 \ldots x_n = v_n} = f(v_1 \ldots v_n)$$

for any $v_1 \ldots v_n \in \{T, F$

$$f(v_1 \ldots v_n) \stackrel{def}{=} (v_1 \wedge v_2) \vee v_3$$

$$\varphi = (x_1 \wedge x_2) \vee x_3$$

Let $\Omega = \{x_1 \ldots x_n\}$. A literal on Ω is either x_i or $\neg x_i$

Let $u \in \{T, F\}$. Then $x_i^u = \begin{cases} x_i & \text{if } u = T \\ \neg x_i & \text{if } u = F \end{cases}$

is a literal.

Let $v \in \{T, F\}^n$. Consider $\bigwedge_{i=1}^{n} x_i^{v_i} = \varphi_v$

this is jus a temp. not.

Let's illustrate this.

Assume $n = 2$ and $v = (T, F)$

$x_1^{v_1} = x_1 \quad x_2^{v_2} = \neg x_2 \quad \varphi_v = x_1 \wedge \neg x_2$

Claim For any $u \in \{T, F\}^n$, $\varphi_v \big|_{x_1 = u_1, \ldots x_n = u_n} = T$ iff $u = v$

Note that

$$f(v_1 \dots v_n) = \left(\bigvee_{\substack{u \in \{T, F\}^n : \\ f(u) = T}} \varphi_u \right) \Bigg|_{x_1 = v_1 \dots x_n = v_n}$$

x_i^u is a statement saying that x_i is equal to u

Definition

A formula φ is in disjunctive normal form if $\varphi = \bigvee \bigwedge l$ (i.e. it's a disj of conj. of literals)

A conj. of literals is called a term

A formula φ is in conj normal form if
$\varphi = \bigwedge \bigvee \ell$

And a disjunctions of literals is a clause.

Exercis

We showed that any Boolean function has a DNF show that it has a CNF.

$$\neg\big((x \wedge y) \vee (y \wedge z)\big) \rightsquigarrow \big(\neg(x \wedge y)\big) \wedge \big(\neg(y \wedge z)\big) \rightsquigarrow$$

$$\rightsquigarrow \underbrace{(\neg x \vee \neg y) \wedge (\neg y \vee \neg z)}_{\text{CNF}}$$

Let's fix some $f : \{T, F\}^n \to \{T, F\}$

We showed that there are $\underbrace{l_{ij}\text{s}}_{\text{literals}}$ s.t.

$$\neg f(v_1 \ldots v_n) = \left(\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} l_{ij} \right) \Bigg|_{x_1 = v_1 \ldots x_n = v_n}$$

Note that

$$\left(\neg \left(\bigvee \bigwedge l_{ij} \right) \right) \Bigg|_{x_1 = v_1 \ldots x_n = v_n} =$$

$$= \left(\bigwedge \bigvee (\neg l_{ij}) \right) \Bigg|_{x_1 = v_1 \ldots x_n = v_n}$$

$$\underbrace{\neg\neg f(v_1 \ldots v_n)}_{\substack{\shortparallel \\ f(v_1 \ldots v_n)}} = \left(\bigwedge \bigvee \neg l_{ij} \right) \Bigg|_{x_1 = v_1 \ldots x_n = v_n}$$