Name: ______________________

Pid: ______________________

Note that since this class is about proofs, every statement in the final exam should be proved. The only exceptions are statements that were proven in previous homework or midterms and statements proven earlier in the class.

1. (20 points) Check all the correct statements.
   - ◯ For any positive integer n, $|[n]^2| > |[n]|$.
   - ◯ If you have 15 balls in 5 boxes, then there is a box with at least 3 balls.
   - ◯ There are 12 elements in the set $[6] \cup \{x : 12 < x < 19\}$.
   - ◯ There are 10 ways to select 2 objects out of 3.
   - ◯ The set $[2]^{[3]}$ has 8 elements.
   - ◯ The function $f : [2] \to [3]$ such that $f(x) = x$ belongs to the set $[2]^{[3]}$.
   - ◯ The function $-x$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$.
   - ◯ $p \wedge \neg p$ is always true.
   - ◯ There is an injection from $[5] \times [5]$ to $[25]$.
   - ◯ A function $f : X \to Y$ is an injection iff the set $\{x \in X : f(x) = y\}$ has cardinality at most 1 for all $y \in Y$.

2. (10 points) Prove the following recurrent formula:

$$S(n, k) = k \cdot S(n-1, k-1) + k \cdot S(n-1, k),$$

where $S(n, k)$ denotes the number of surjective functions from $[n]$ to $[k]$.

3. (10 points) Show that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$ for any positive integer $n \geq 2$.

4. (10 points) In school there are three clubs and for any two students there is at least one club such that both of them are in this club. Show that for some club $2/3$ of the students are in this club.

5. (10 points) We say that a function $f : \{0,1\}^n \to \{0,1\}$ depends on the ith argument iff for some $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in \{0,1\}$

$$f(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n) \neq f(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n).$$

We also say that the function f depends on all the arguments iff for all $i \in [n]$ it depends on ith argument.

Find the number of functions $f : \{0,1\}^n \to \{0,1\}$ depending on all arguments.

6. (10 points) How many integer numbers from 0 to 999 are having at least one digit equal to 7.

7. (10 points) Let us consider Young's geometry, it is a theory with undefined terms: point, line, is on, and axioms:

    1. there exists at least one line,
    2. every line has exactly three points on it,
    3. not all points are on the same line,
    4. for two distinct points, there exists exactly one line on both of them,
    5. if a point does not lie on a given line, then there exists exactly one line on that point that does not intersect the given line.

    Show that for every point, there are exactly four lines on that point.