Chapter 8

Bijections, Surjections, and Injections

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Bijections, Surjections, and Injections

In the previous chapters we used the property that the set is finite. However, we have never defined formally what it means. In this chapter we define cardinality which is a formalization of the notion size of the set.

8.1 Bijections

Definition 8.1. *Let $f : X \to Y$ be a function. We say that f is a bijection iff the following properties are satisfied.*

- *Every element of Y is an image of some element of X. In other words,*

$$\forall y \in Y \ \exists x \in X \ f(x) = y.$$

- *Images of any two elements of X are different. In other words,*

$$\forall x_1, x_2 \in X \ f(x_1) \neq f(x_2).$$

Let us consider the following example. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = x + 1$; Note that it is a bijection:

- If $f(x_1) = f(x_2)$, then $x_1 + 1 = x_2 + 1$ i.e. $x_1 = x_2$.
- For any $y \in \mathbb{R}$, $f(y - 1) = (y - 1) + 1 = y$.

Exercise 8.1. *Show that x^3 is a bijection.*

One of the nicest properties of bijections is that composition of two bijections is a bijection.

Theorem 8.1. *Let X, Y, and Z be some sets and $f : X \to Y$ and $g : Y \to Z$ be bijections. Then $(g \circ f) : X \to Z$ is also a bijection.*

Proof. We need to check two properties.

- Let $x_1 \neq x_2 \in X$. Note that $f(x_1) \neq f(x_2)$ since f is a bijection. Hence, $g(f(x_1)) \neq g(f(x_2))$ since g is a bijection as well. As a result, $(g \circ f)(x_1) \neq (g \circ f)(x_2)$.
- Let $z \in Z$; we need to find $x \in X$ such that $(g \circ f)(x) = y$. Note that since g is a bijection there is $y \in Y$ such that $g(y) = z$. Additionally, there is $x \in X$ such that $f(x) = y$ since f is a bijection. Thus, $(g \circ f)(x) = g(f(x)) = z$.

□

Probably the most important property of a bijection is that we may invert it.

Theorem 8.2. *Let $f : X \to Y$ be a function. f is invertible (i.e. there is a function $g : Y \to X$ such that $(f \circ g)(y) = y$ and $(g \circ f)(x) = x$ for all $x \in X$ and $y \in Y$) iff f is a bijection.*

Proof. $\Rightarrow$ Let's assume that f is invertible. We need to prove that f is a bijection.

- Let's assume that f does not satisfy the first property in the definitions of bijections i.e. there are $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$ but $x_1 = g(f(x_1)) = g(f(x_2)) = x_2$, which is a contradiction.
- Let $y \in Y$. Note that $f(g(y)) = y$, hence, $\mathrm{Im} f = Y$.

$\Leftarrow$ Let's assume that f is bijective. We need to define a function $g : Y \to X$ which is an inverse of f. Let $y \in Y$, note that there is a unique x such that $f(x) = y$, we define $g(y) = x$. Note that $f(g(y)) = y$ for every y by the construction of g. Additionally, $g(f(x)) = x$ since $f(g(f(x))) = f(x)$ and f is a bijection.

□

One may notice that if we have a bijection f from $[n]$ to a set S we enumerate all the elements of S: $f(1), \dots, f(n)$. This observation allows us to define the cardinality of a set.

Definition 8.2. *Let S be a set, we say that cardinality of S is equal to n (we write that $|S| = n$) iff there is a bijection from $[n]$ to S.*

We also say that a set T is finite if there is an integer n such that $|T| = n$.

Note that this definition does not guarantee that cardinality is unique.

Theorem 8.3. *For any set S, if there are bijections $f : [n] \to S$ and $g : [m] \to S$, then $n = m$.*

Proof. Let us consider the inverse g^{-1} of g. Note that $h = f \circ g^{-1}$ is a bijection from $[n]$ to $[m]$.

We prove using induction by n that for any $n, m \in \mathbb{N}$, if there is a bijection h' from $[n]$ to $[m]$, then $n = m$. The base case is for $n = 1$; if $m \geq 2$, then there are $x, y \in [1]$ such that $h'(x) = 1$ and $h'(y) = 2$, but $x \neq y$ and we have only one element in $[1]$.

The induction step is also simple. Assume that there is a bijection h' from $[n+1]$ to $[m]$. We define a function $h'' : [n] \to [m-1]$ as follows:

$$h''(i) = \begin{cases} h'(i) & \text{if } h'(i) < h(n+1) \\ h'(i) - 1 & \text{otherwise} \end{cases}.$$

We prove that h'' is a bijection.

- Let $i_1 \neq i_2 \in [n]$. If $h'(i_1), h'(i_1) < h'(n+1)$ or $h'(i_1), h'(i_1) \geq h'(n+1)$, then $h''(i_1) \neq h''(i_2)$ since $h'(i_1) \neq h'(i_2)$. Otherwise, without loss of generality we may assume that $h'(i_1) < h(n+1) < h'(i_2)$ but it implies that $h''(i_1) = h'(i_1) < h'(n+1) \leq h'(i_2) - 1 = h''(i_2)$.
- Let $j \in [m-1]$. We need to consider two cases.
  1. Let $j < h(n+1)$. There is $i \in [n+1]$ such that $h'(i) = j$ since h' is a bijection (note that $i \neq n+1$). Thus $h''(i) = j$.
  2. Otherwise, there is $i \in [n+1]$ such that $h'(i) = j + 1$ since h' is a bijection (note that $i \neq n+1$). Thus $h''(i) = j$.

Since h'' is a bijection, the induction hypothesis implies that $n = m - 1$. As a result, $n + 1 = m$. □

Using Theorem 8.2 we may derive a way to apply this theory in combinatircs; we can use a bijection to prove that two sets have the same cardinality.

Theorem 8.4. *Let X and Y be two finite sets such that there is a bijection f from X to Y. Then $|X| = |Y|$.*

Proof. Let $|X| = n$, and $g : [n] \to X$ be a bijection. Note that $f \circ g : [n] \to Y$ is a bijection, hence $|Y| = m$. □

Using this result we can make prove the following equality.

Corollary 8.1. *Let X be a finite set of cardianlity n. Then 2^X has the same cardinality as $\{0,1\}^{|X|}$.*

Proof. To prove this statement we need to construct a bijection from 2^X to $\{0,1\}^{|X|}$. Let $|X| = n$ and $f : [n] \to X$ be a bijection.

First we construct a bijection g_1 from 2^X to $2^{[n]}$: $g_1(Y) = \{f(x) \; : \; x \in Y\}$ ($Y \in 2^X$). It is easy to see that the function $g_1^{-1}(Y) = \left\{f^{-1}(x) \; : \; x \in [n]\right\}$ ($Y \in 2^{[n]}$) is an inverse of g_1, so g_1 is indeed a bijection.

Now we need to construct a bijection g_2 from $2^{[n]}$ to $\{0,1\}^n$: $g_2(Y) = (u_1, \ldots, u_n)$, where $u_i = 1$ iff $i \in Y$. It is clear that $g_2^{-1}(u_1, \ldots, u_n) = \{i \in [n] \ : \ u_i = 1\}$ is an inverse of g_2 so g_2 is indeed a bijection.

As a result, by Theorem 8.1, the function $(g_2 \circ g_1) : 2^X \to \{0,1\}^{|X|}$ is a bijection. □

Theorem 8.5. *Let X, Y, Z be some sets There are bijctions from $X \times (Y \times Z)$ and $(X \times Y) \times Z$ to $\{(x, y, z) \ : \ x \in X, y \in Y, z \in Z\}$.*

Proof. Since the statement is symmetric, it is enough to prove that there is a bijection f from $X \times (Y \times Z)$ to $\{(x, y, z) \ : \ x \in X, y \in Y, z \in Z\}$. Define f such that $f(x, (y, z)) = (x, y, z)$. Clearly, $f^{-1}(x, y, z) = (x, (y, z))$ is the inverse of f, so f is indeed a bijection. □

8.2 Surjections and Injections

It is possible to note that the definition of the bijection consists of two part. Both of these parts are interesting in their own regard, so they have their own names.

Definition 8.3. *Let $f : X \to Y$ be a function.*

- *We say that f is a surjection iff every element of Y is an image of some element of X. In other words,*

$$\forall y \in Y \ \exists x \in X \ f(x) = y.$$

- *We say that f is an injection iff images of any two elements of X are different. In other words,*

$$\forall x_1, x_2 \in X \ f(x_1) \neq f(x_2).$$

Remark 8.1. *Let $f : X \to Y$ be an injection. Then $g : X \to \mathrm{Im} f$ such that $f(x) = g(x)$ is a bijection.*

Exercise 8.2. *Let $\mathbb{R}^+ = \{x \in \mathbb{R} \ : \ x > 0\}$. Is $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(x) = x + 1$ a surjection/injection?*

Like in the case of the bijection we may use surjections and injections to compare sizes of sets.

Theorem 8.6. *Let X and Y be finite sets.*

- *If there is an injection from X to Y, then $|X| \leq |Y|$.*
- *If there is a surjection from X to Y, then $|X| \geq |Y|$.*

8.3 Generalized Commutative Operations

Using the notation of cardianlity we may generalize the summation operation in the followin way: $\sum_{i \in S \ : \ P(i)} f(i)$ is equal to the sum of $f(i)$ for all the $i \in \{i \in S \ : \ P(i)\}$; i.e.

$$\sum_{i \in S \ : \ P(i)} f(i) = \sum_{j=1}^{k} f(i_j),$$

where $\{i \in S \ : \ P(i)\} = \{i_1, \ldots, i_k\}$. More formally,

$$\sum_{i \in S \ : \ P(i)} f(i) = \sum_{j=1}^{k} f(g(j)),$$

where $k = |\,\{i \in S \ : \ P(i)\}\,|$ and $g : \{i \in S \ : \ P(i)\} \to [k]$ is a bijection.

Theorem 8.7. *The definition of* $\sum_{i \in S : P(i)} f(i)$ *is correct; i.e.* $\sum_{i=1}^{k} f(g_1(i)) = \sum_{i=1}^{k} f(g_2(i))$ *for any two bijections* $g_1, g_2 : \{i \in S \ : \ P(i)\} \to [k]$.

Proof. Proof of this theorem consists of two parts. First, we prove that

$$\sum_{i=1}^{k} f(g(i)) = \sum_{i=1}^{k} f(g(h(i))) \tag{8.1}$$

for any bijections $g : \{i \in S \ : \ P(i)\} \to [k]$ and $h : [k] \to [k]$.

To prove this statement, we introduce the notion of inversion. We say that $i, j \in [k]$ for an inversion in h iff $h(i) > h(j)$ and $i < j$. We denote by $I(h)$ the number of inversions in h; i.e. $I(h) = |\,\{(i,j) \ : \ i,j \text{ form an inversion in } h\}\,|$. It is easy to see that $I(h) = 0$ iff $h(i) = i$ for any $i \in [k]$. It is also clear that if i, j form an inversion in h, then $I(h) > I(h')$, where

$$h'(x) = \begin{cases} h(j) & \text{if } x = i \\ h(i) & \text{if } x = j \\ h(x) & \text{otherwise} \end{cases}.$$

We prove Equation 8.1 using the induction by $I(h)$.

The base case: if $I(h) = 0$, then h is an identity function and $g(i) = g(h(i))$. Hence, Equation 8.1 is true.

The induction step: by the induction hypothesis, if $I(h') < \ell$, then

$$\sum_{i=1}^{k} f(g(i)) = \sum_{i=1}^{k} f(g(h'(i)))$$

for any bijection $h' : [k] \to [k]$. Let us consider a bijection $h : [k] \to [k]$ such that $I(h) = \ell$. Define $h' : [k] \to [k]$ such that

$$h'(x) = \begin{cases} h(j) & \text{if } x = i \\ h(i) & \text{if } x = j \\ h(x) & \text{otherwise} \end{cases}.$$

Note that by the induction hypothesis,

$$\sum_{i=1}^{k} f(g(i)) = \sum_{i=1}^{k} f(g(h'(i)))$$

and it is clear that

$$\sum_{i=1}^{k} f(g(h'(i))) = \sum_{i=1}^{k} f(g(h(i))).$$

As a result, Equation 8.1 is true.

Now we are ready to finish proof of the theorem. Consider $g_1, g_2 : \{i \in S \ : \ P(i)\} \to [k]$ and define $h = g_1^{-1} \circ g_2$. Note that $h : [k] \to [k]$ is a bijection and $g_1(h(i)) = g_2(i)$. Thus we proved that

$$\sum_{i=1}^{k} f(g_1(i)) = \sum_{i=1}^{k} f(g(h(i))) = \sum_{i=1}^{k} f(g_2(i)).$$

□

Similarly one may define a generalized union and intersection of sets. Let Ω and S be some sets, $X : S \to 2^{\Omega}$ and $P(i)$ be a predicate. Then

$$\bigcup_{i \in S \ : \ P(i)} X(i) = \bigcup_{i=1}^{k} X(g(i)$$

and

$$\bigcap_{i \in S \ : \ P(i)} X(i) = \bigcap_{i=1}^{k} X(g(i),$$

where $k = |\{i \in S \ : \ P(i)\}|$ and $g : \{i \in S \ : \ P(i)\} \to [k]$ is a bijection.

Exercise 8.3. *Show that the definitions of* $\bigcup_{i \in S : P(i)} X(i)$ *and* $\bigcap_{i \in S : P(i)} X(i)$ *are correct, i.e. that the do not depend on the choice of* g.

End of The Chapter Exercises

8.4 Construct a bijection from $\{0, 1, 2\}^n$ to

$$\{(A, B) \ : \ A, B \subseteq [n] \text{ and } A, B \text{ are disjoint}\}.$$

8.5 Construct a bijection from $\{0, 1\} \times [n]$ to $[2n]$.

8.6 Prove Theorem 8.6.