Name: ______________________

Pid: ______________________

1. (10 points) We say that L is a B-decision list

   (base case) if either L is a number $y \in \mathbb{Z}$, or

   (recursion step) L is equal to (f, v, L') where $f : \mathbb{Z} \to \{0, 1\}$, $v \in \mathbb{Z}$, and L is a B-decision list.

   We can also define the value $\mathrm{val}(L, x)$ of a B-decision list L at $x \in \mathbb{Z}$.

   (base case) If L is a number y, then $\mathrm{val}(L, x) = y$, and

   (recursion step) if $L = (f, v, L')$, then

$$\mathrm{val}(L, x) = \begin{cases} v & \text{if } f(x) = 1 \\ \mathrm{val}(L', x) & \text{otherwise} \end{cases}.$$

   Similarly one may define the length $\ell(L)$ of a B-decition list L.

   (base case) If L is a number y, then $\ell(L) = 1$, and

   (recursion step) if $L = (f, v, L')$, then $\ell(L) = \ell(L') + 1$.

   Assume that $\mathrm{val}(L, x) = x$ for any $x \in [1000]$ show that $\ell(L) \geq 1000$.

Solution: For a B-decision list L, we define $V(L) = \{\mathrm{val}(L, x) \ : \ x \in \mathbb{Z}\}$.

We prove using structural induction that the size of $V(L)$ is at most $\ell(L)$.

Let S' be the set of B-decition lists such that the size of $V(L)$ is at most $\ell(L)$.

- Note that if L is a number y, then $\mathrm{val}(L, x) = y$ for all $x \in \mathbb{Z}$; therefore $L \in S'$.
- Assume $L' \in S'$ and $L = (f, v, L')$. It is clear that $V(L) \subseteq V(L') \cup \{v\}$. Therefore the size of $V(L)$ is at most $\ell(L') + 1 = \ell(L)$.

As a result, by the structural induction theorem, $S' = S$. Which means that the size of $V(L)$ is at most $\ell(L)$.

Assume that $\mathrm{val}(L, x) = x$ for any $x \in [1000]$. This implies that $V(L) \geq 1000$; hence, $\ell(L) \geq 1000$ by the previous observation.

2. (10 points) Let S be the minimal set such that $3 \in S$ and $(x+y) \in S$ for any $x, y \in S$. (In other words, S is generated by $\{f\}$ from $\{3\}$, where $f(x,y) = x+y$.) Show that $S = \{3k \ : \ k \in \mathbb{N}\}$.

Solution: The statement consists of two parts: $S \subseteq \{3k \ : \ k \in \mathbb{N}\}$ and $S \supseteq \{3k \ : \ k \in \mathbb{N}\}$.

- Note that $\{3\} \subseteq \{3k \ : \ k \in \mathbb{N}\}$ and $f(3k, 3\ell) = 3k + 3\ell = 3(k+\ell)$. Therefore, by the principle of structural induction $S \subseteq \{3k \ : \ k \in \mathbb{N}\}$.
- We prove using induction by k that $3k \in S$ for all $k \in \mathbb{N}$. The base case for $k = 1$ is true since $3 \in S$. Let us prove the induction step from k to $k+1$. Assume that $3k \in S$; then $f(3k, 3) = 3(k+1) \in S$ as well. As a result, by the induction principle, $3k \in S$ for all $k \in \mathbb{N}$; i.e., $S \supseteq \{3k \ : \ k \in \mathbb{N}\}$.