Name: ________________

Pid: ________________

Show all of your work. Full credit will be given only for answers with explanations.

1. (50 points) Check all the correct statements.

   ◯ The tangent plane of the funciton $f(x,y) = xe^y + ye^x$ at $(1,1,2e)$ is defined by the equation
   $$2ex + 2ey = z + 2e.$$

   ◯ The angle between $\frac{\partial f(\pi,0)}{\partial x}$ and $\frac{\partial f(\pi,0)}{\partial y}$ is $\pi/2$, where $f(x,y) = \langle \cos(x)+\sin(y), \sin(x)+\cos(y)\rangle$.

   ◯ If $z = x^2 + y^2$, $x = \sin(t)$, and $y = \cos(t)$, then $\frac{dz}{dt} = 0$

   ◯ The tangent planes of $f(x,y) = x^2 + y^2$ at $(1,0,1)$ and $(0,1,1)$ are parallel.

   ◯ The vector $\langle 1,-1,1\rangle$ is perpendicular to $\frac{df(\pi)}{dt}$ and $\frac{df(\pi/2)}{dt}$, where $f(t) = \langle \cos(t), \sin(t), t\rangle$.

Solution:

- First of all we need to compute the partial derivatives of f, $\frac{\partial f}{\partial x} = e^y + ye^x$ and $\frac{\partial f}{\partial y} = xe^y + e^x$. If $x = 1$ and $y = 1$, then $\frac{\partial f}{\partial x} = 2e$ and $\frac{\partial f}{\partial y} = 2e$. As a result the tangent plane is $2e(x-1) + 2e(y-1) = z - 2e$ which can be simplified to $2ex + 2ey = z + 2e$.
- Let us first find explicitly the derivatives, $\frac{\partial f}{\partial x} = \langle -\sin(x), \cos(x)\rangle$ and $\frac{\partial f}{\partial y} = \langle \cos(y), -\sin(x)\rangle$. At point $(\pi,0)$ the value of the derivatives are equal to $\langle 0,-1\rangle$ and $\langle 1,0\rangle$. The angle between these vectors is $\frac{\pi}{2}$.
- Note that $x^2 + y^2 = \cos^2(t) + \sin^2(t)$ which is equal to 1 for any t. Hence, $z = 1$ and $\frac{dz}{dt} = 0$.
- We need to find the partial derivatives of f, $\frac{\partial f}{\partial x} = 2x$ and $\frac{\partial f}{\partial y} = 2y$. Hence, the tangent planes for this surface are $z - 1 = 2(x-1)$ and $z - 1 = 2(y-1)$ and they are not parallel.
- Let us find the derivative if f, $\frac{df}{dt} = \langle -\sin(t), \cos(t), 1\rangle$. Hence, $\frac{df(\pi)}{dt} = \langle 0,-1,1\rangle$ and $\frac{df(\pi/2)}{dt} = \langle -1,0,1\rangle$. Since $\langle 1,-1,1\rangle \cdot \langle 0,-1,1\rangle = 2$ they are not perpendicular.

2. Let $r = \langle x^y, y^x \rangle$, where $x = e^t$, and $y = t^2$.

(a) (5 points) Find $\frac{dr}{dt}$.

Solution: First of all, we need to find $\frac{dx^y}{dt}$ and $\frac{dy^x}{dt}$.

$$\frac{dx^y}{dt} = \frac{\partial x^y}{\partial x}\frac{dx}{dt} + \frac{\partial x^y}{\partial y}\frac{dy}{dt} = (yx^{y-1}) \cdot e^t + (x^y \ln x) \cdot 2t$$

$$\frac{dy^x}{dt} = \frac{\partial y^x}{\partial x}\frac{dx}{dt} + \frac{\partial y^x}{\partial y}\frac{dy}{dt} = (y^x \ln y) \cdot e^t + (xy^{x-1}) \cdot 2t$$

As a result,

$$\frac{dr}{dt} = \langle (yx^{y-1}) \cdot e^t + (x^y \ln x) \cdot 2t, (y^x \ln y) \cdot e^t + (xy^{x-1}) \cdot 2t. \rangle$$

(b) (5 points) Find the tangent line of the curve described by the vector function r for $t = 1$

Solution: If $t = 1$, then $x = e$, $y = 1$, and $r = \langle e, 1 \rangle$. Hence, $\frac{dr}{dt} = \langle 1 \cdot 1 \cdot e^1 + (e \cdot 1) \cdot 2, 1 \cdot 0 \cdot e + (1 \cdot e) \cdot 2 \rangle = \langle 3e, 2e \rangle$.

As a result, the answer is the line going throw $\langle e, 1 \rangle$ with the slope $\frac{2e}{3e}$. Hence, the answer is $y = \frac{2}{3}(x - e) + 1$.

3. Let $f(x, y) = xy^2 + yx^2$.

   (a) (5 points) Find the tangent planes to the surface defined by f at $(1, 1, 2)$ and $(-1, -1, -2)$.

   Solution:

   (b) (5 points) Check if these planes are intersectin; if they are intersecting, find symmetric equations for the line of intersection of the planes.

   Solution:

4. Let us consider a surfece defined implicitly by the equation $x^3 + y^3 + z^3 + 6xyz = 1$. Find the tangent plane of the surface at $(1, -3, -3)$.

Solution: Let us compute the partial derivative by x of both sides of the equality

$$x^3 + y^3 + z^3 + 6xyz = 1$$

, we get $3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} = 0$. As a result, $\frac{\partial z}{\partial x} = -\frac{3x^2+6yz}{3z^2+6xy}$. Similarly, $\frac{\partial z}{\partial y} = -\frac{3y^2+6xz}{3z^2+6xy}$.

Since $x = 1$, $y = -3$, and $z = -3$, we may find the values of the partial derivatives: $\frac{\partial z}{\partial x} = -\frac{3+6\cdot 9}{27-6\cdot 3} = -\frac{57}{9}$ and $\frac{\partial z}{\partial y} = -\frac{27-6\cdot 3}{27-6\cdot 3} = -1$.

Therefore, the tangent plane to the surface is defined by the equation $z + 3 = -\frac{57}{9}(x-1) - (x+3)$.