Name: ____________________

Pid: ____________________

1. Prove that $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2$.

Solution: Let us prove a stronger statement:

$$1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

The base case is clear. We prove now the induction step. By the induction hypothesis the following inequality holds

$$1 + \frac{1}{2^2} + \cdots + \frac{1}{(n-1)^2} \leq 2 - \frac{1}{n-1}.$$

Hence,

$$1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n-1} + \frac{1}{n^2} = 2$$

but $\frac{1}{n-1} - \frac{1}{n^2} \geq \frac{1}{n}$. Indeed, $\frac{1}{n-1} \geq \frac{n+1}{n^2}$ is equivalent to $\frac{1}{(n-1)^2} \geq \frac{1}{n^2}$ wcich is true. As a result we proved the induction step.

2. Show that $(1+x)^n \geq 1+nx$ for every $n \in \mathbb{N}$ and $x \geq -1$.

Solution: We are going to prove that by induction by n.

The induction step is clear. We prove now the induction step. Assume the induction hypothesis: $(1+x)^n \geq 1+nx$. Note that $(1+x)^{n+1} \geq (1+nx) \cdot (1+x) = 1+nx+x+nx^2 \geq 1+(n+1)x$. Which proves the induction step.

3. There are irrational numbers a and b such that a^b is rational.

Solution: Assume that a^b is irrational for all irrationals a and b. Since $\sqrt{2}$ is irrational $\sqrt{2}^{\sqrt{2}}$ is also irrational. However, $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$, which contradicts to the assumption.
As a result, there are irrational a and b such that a^b is rational.

4. If $a, b \in \mathbb{Z}$, then $a^2 - 4b + 2 \neq 0$.

Solution: Assume, for the sake of contradiction, that such a and b exist. Note that $a^2 = 4b + 2$, hence, a is even but a is not divisible for 4 which is a contradiction.